Optimisation and sensitivity analysis of GPS receiver tracking loops in dynamic environments

D.-J. Jwo

Abstract: For a GPS receiver, decreasing the receiver tracking loop bandwidth reduces the probability of loss of lock if there are no vehicle dynamics. However, reduced bandwidth increases tracking errors due to dynamics. Beyond a certain limit it causes a serious degradation in the dynamic tracking performance. Therefore, there is involvement of a tradeoff between two opposing considerations: narrow tracking loop bandwidths are desired for filtering noise due to thermal effects, but wide tracking loop bandwidths are desired to permit tracking of vehicle dynamics. Optimal tracking loop bandwidths, which yield the minimum errors in a certain dynamics environment, are first investigated. The linear Kalman filter is employed as the optimal estimator. The covariance for the arbitrary gain model is solved and applied to the sensitivity analysis for investigating error growth due to incorrect noise level estimate. Theoretical results are verified by numerical simulation, and results from both approaches are in very good agreement.

1 Introduction

The tracking errors of a receiver operating on the GPS code and carrier include two major components: noise error, caused by thermal noise; and transient error, caused by imperfectly tracking the vehicle dynamics. Selection of the baseband processor design for a GPS receiver always involves a tradeoff between two opposing considerations. Narrow tracking-loop bandwidths are desired for filtering noise due to thermal effects or jamming, however, wide tracking-loop bandwidths are desired to permit tracking signal Doppler shifts induced by vehicle/user dynamics.

Either the carrier loop or the code loop is usually designed to select a bandwidth which produces tracking errors under maximum dynamics approximately equal to the lock limit of the loop. When the GPS signal power is limited, the tendency would seem to make the receiver tracking-loop bandwidth narrower. However, this increases the probability that the tracking loop will lose lock owing to vehicle/user dynamics. Thus, there is a fundamental system limitation of tracking-loop threshold when considering both low carrier-to-noise ratio and user dynamics at the same time. It is therefore important to analyse the error characteristics for determining the optimal loop bandwidth that minimises the total tracking error.

Receiver noise models for predicting the thermal noise jitters have been presented, for example [1–7]. Dynamics stress errors can also be accurately predicted [4, 6, 7]. The summation of these two major error components has a minimum value for certain carrier-to-noise ratio and user dynamics. Based on knowledge of the errors, the theoretical prediction of optimal bandwidth for minimum tracking error can be determined. The environment concerned is the case of limited GPS signal power or jamming for the dynamics user without other information aiding. How the incorrect parameters (i.e. departure from the design point: could be intentional or unintentional) influence the error growth, referred to as the ‘sensitivity analysis’, is considered. The sensitivity analysis involves the incorrect estimate of received signal carrier-to-noise ratio. A theoretical approach and numerical simulation are performed for verification.

2 GPS receiver tracking loops

The GPS receiver contains a code tracking loop and a carrier tracking loop for tracking the Doppler-shifted carrier. The pseudorange obtained from the code tracking loop provides a position fix; the pseudorange rate estimate obtained from the carrier tracking loop provides a velocity fix. The receiver carrier tracking loop is more sensitive to dynamics due to the fact that it tracks a much higher frequency signal than a code tracking loop. If the carrier tracking loop loses lock during a dynamic manoeuvre, the code tracking loop will usually lose lock subsequently.

Some designs use the carrier loop to track the dynamics and provide the code loop with a prior knowledge of the dynamics such that the code loop will not see the full dynamics. The external navigation source, such as inertial velocity, can also be utilised to aid the tracking loops for removing most of the dynamics stress error such that a smaller bandwidth could be used. More information regarding the inertial velocity aiding can be found in [9, 10]. The architecture of the simplified GPS receiver tracking loops is shown in Fig. 1.

2.1 Transfer functions of tracking loops

A generalised block diagram of a tracking loop that is applicable for analysis of both carrier and code loops is
shown in Fig. 2. The closed-loop transfer function of the tracking loop is

$$H(s) = \frac{\hat{\theta}(s)}{\theta(s)} = \frac{K_g K_f F(s)}{s + K_a K_g F(s)} = \frac{G(s)}{1 + G(s)} \tag{1}$$

where the open-loop transfer function is

$$G(s) = \frac{K_g K_f F(s)}{s} \tag{2}$$

$K_g$ represents the voltage-controlled oscillator (VCO) gain factor, $K_f$ is the phase-detector/delay discriminator gain factor. The transfer function $F(s)$ represents the loop filter of a tracking loop. The loop filter is usually a low-pass filter used to suppress the noise and high-frequency signal components from the phase-detector/delay discriminator and provide a DC-controlled signal for the VCO. A first-order tracking loop is obtained when $F(s) = 1$. There are usually three options for selecting the loop filters in a second-order loop: (a) simple lag filter, (b) active filter or (c) passive filter. A second-order loop with lag filter is referred to as a modified first-order loop rather than a genuine second-order loop. The second-order tracking loop with passive filter when $1/K_a \ll \tau$ will be nearly the same as the one with active filter. Third orders are insensitive to acceleration, and optimal for constant jerk (rate of change of acceleration) input; fourth-order loops are insensitive to jerk. It is rare that a loop is constructed with an order higher than third. More information on tracking loop related topics is provided in [7].

### 2.2 Equivalent noise bandwidth

The single-side equivalent noise bandwidth, in Hz, for a tracking loop with transfer function $H(j\omega)$ is expressed as

$$B_n = \frac{1}{|H(0)|^2} \int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega \tag{3}$$

where $\omega = 2\pi f$ and the magnitude of the frequency response is

$$|H(j\omega)|^2 = [H(j\omega)H(-j\omega)] \tag{4}$$

Analytical solution for the tracking loop bandwidths, by implementing eqn. 3, might be tedious. R. S. Philips’ table of contour integrals can be employed for simpler computation

$$I_n = \frac{1}{2\pi f} \int_{-\infty}^{\infty} c(s)c(-s) \frac{ds}{a(s)a(-s)} \tag{5}$$

where

$$c(s) = c_{-1}s^{-1} + c_{-2}s^{-2} + \cdots + c_0$$

$$a(s) = a_0 + a_1 s^{-1} + a_2 s^{-2} + \cdots + a_0$$

Fig. 3 provides solutions for such type of integration up to fourth order. Solutions for higher orders can be found in [8]. Fig. 4 provides the loop filters, closed-loop transfer functions, and equivalent noise bandwidths, for different orders of receiver tracking loops.

### 3 Receiver error behaviour

The tracking errors of the receiver operating on the GPS code and carrier have two major components: noise error caused by thermal noise, and transient error caused by imperfectly tracking the user dynamics.

#### 3.1 Thermal noise

Closed-form expressions for the tracking error due to thermal noise are derived under the assumption of infinite signal bandwidth [3]. The one-sigma errors of thermal noises with regard to different types of code tracking loops are as follows:

**Coherent**

$$\sigma_{D,L,L} \approx \left[ \frac{B_1 d}{2c/n_0} \right]^{1/2} \tag{6a}$$

**Early-minus-late power**

$$\sigma_{D,L,L} = \left[ \frac{B_1 d}{2c/n_0} \left( 1 + \frac{2}{T(2-d)c/n_0} \right) \right]^{1/2} \tag{6b}$$

**Dot-product**

$$\sigma_{D,L,L} = \left[ \frac{B_1 d}{2c/n_0} \left( 1 + \frac{1}{Tc/n_0} \right) \right]^{1/2} \tag{6c}$$

where $\sigma_{D,L,L}$ is the standard deviation of tracking error in units of PRN chips, $B_1$ is the single-sided code tracking loop bandwidth in Hz, $T$ is the predetection integration time.
\[ I_n = \frac{1}{2\pi n} \int_{-\infty}^{\infty} c(x)c(-x) \frac{a(x)a(-x)}{ds} \, dx, \quad n = 1, 2, 3, \ldots \]
\[ c(s) = c_{s-1}s^{n-1} + c_{s-2}s^{n-2} + \cdots + c_0 \]
\[ a(s) = a_{s+1}s^{n+1} + \cdots + a_0 \]
\[ I_1 = \frac{c_0^2}{2a_0a_1} \]
\[ I_2 = \frac{c_1^2a_0 + c_2^2a_2}{2a_0a_1a_2} \]
\[ I_3 = \frac{c_1^2a_0a_1 + (c_1^2 - 2c_1c_2)a_0a_3 + c_2^2a_2a_3}{2a_0a_2(a_1a_2 - a_0a_3)} \]
\[ I_4 = \frac{c_1^2(-a_0^2a_1 + a_0a_3a_4) + (c_1^2 - 2c_1c_2)a_0a_4 + (c_1^2 - 2c_1c_2)a_0a_4 + c_2^2(-a_1^2a_4 + a_2a_3a_4)}{2a_0a_4(-a_0^2a_1 - a_1^2a_4 + a_2a_3a_4)} \]

Fig. 3 List of contour integrals

time (PIT) in seconds, \( d \) is the early-to-late correlator spacing normalised with respect to one chip
\[ d = 1 \text{ for time-shared tau-dithered early-late correlator} \]
\[ = \frac{1}{2} \text{ for dedicated early and late correlator} \]
and \( c/n_0 \) is the carrier-to-noise ratio value
\[ c/n_0 = (\text{SNR})(B_L) \text{ (ratio-Hz)} \]
\[ C/N_0 = 10 \log_{10}(c/n_0) \text{ (dB-Hz)} \]
The second term in parentheses is known as the ‘squaring loss’. Most GPS receivers employ a noncoherent delay-lock loop (DLL) for code tracking and a Costas-type phase-lock loop (PLL) for tracking the Doppler-shifted carrier. The thermal noise jitter for the commonly used tau-dithered early-late DLL with \( d = 1 \), is
\[ \sigma_{DLL} = \left[ \frac{B_L}{2c/n_0} \left( 1 + \frac{2}{T_c/n_0} \right) \right]^{1/2} \text{ (chips)} \]
\[ \sigma_{PLL} = \left[ \frac{B_L}{c/n_0} \left( 1 + \frac{1}{2T_c/n_0} \right) \right]^{1/2} \lambda_c \text{ (metres)} \]
The P-code and \( \Lambda \)-code, with the chirping rate (chips per sec) of 10.23 and 1.023 Mbit/s, correspond to a chip width \( \lambda_c \) of 29.305 and 293.05 m, respectively.
The thermal noise for the PLL, when implemented in the form of a Costas-type loop, is approximated by
\[ \sigma_{PLL} = \left[ \frac{B_L}{c/n_0} \left( 1 + \frac{1}{2T_c/n_0} \right) \right]^{1/2} \lambda_c \text{ (metres)} \]

The GPS has a 50 Hz navigation data message bit rate; the predetection integration time is usually the period of a navigation data bit, 20 ms. The wavelengths \( \lambda_i \) for \( L_1 \) and \( L_2 \) carriers, due to the link frequencies 1575.42 and 1227.6 MHz, are 0.193 and 0.2442 m, respectively.

For the DLL, the thermal noise is independent of tracking-loop order; for the PLL, the thermal noise jitter is not directly dependent on the loop order, too [4]. The \( C/N_0 \) of the GPS with good signal power typically range from 35–55 dB-Hz, so tracking errors generally run on the lower end of the range. For the nominal range of \( C/N_0 \) (35 dB-Hz and above) the squaring loss is usually considered negligible. Typically, the unaided code tracking-loop bandwidths are in the range of 1–4 Hz and unaided carrier tracking-loop bandwidths are in the range of 5–15 Hz. Fig. 5 shows one-sigma thermal jitters of the early-minus-late noncoherent DLL and the Costas-type PLL, respectively. The maximum bandwidths allowed for maintaining lock in various \( C/N_0 \) environments are shown in Fig. 6.

### 3.2 Dynamic stress errors

A small steady-state error is usually desired and is considered as the criterion of good tracking performance. The transfer function representing the tracking error is
\[ E(s) = \frac{\theta_e(s)}{\theta(s)} = 1 - H(s) = \frac{s}{s + K_0K_sF(s)} \]

The steady-state error can be evaluated by means of the final value theorem of the Laplace transforms
\[ \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s[1 - H(s)] \]
and thus
\[ \lim_{t \to \infty} \theta_e(t) = \lim_{s \to 0} \frac{s^2\theta(s)}{s + K_0K_sF(s)} \]
The loop order is sensitive to the same order of dynamics, e.g. first order to velocity stress, second order to acceleration stress, and third order to jerk stress. The first-order loop is suitable for a user position that varies in a random walk manner (white noise velocity); the second-order loop is suitable for a user velocity that varies in a random walk manner (white noise acceleration); the third-order loop is suitable for a user acceleration that varies in a random walk manner (white noise jerk). For example, the dynamic condition resulting in a second-order loop tracking error is constant acceleration. There is no steady-state transient.
<table>
<thead>
<tr>
<th>Loop order</th>
<th>1st order</th>
<th>2nd order (active lag–lead)</th>
<th>3rd order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Formulation I: tracking loop description in terms of original circuit parameters</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F(s)$</td>
<td>1</td>
<td>$\frac{\tau_2 s + 1}{\tau_1 s}$</td>
<td>$\left(\frac{\tau_2 + 1}{\tau_1}\right)^2$</td>
</tr>
<tr>
<td>$H(s)$</td>
<td>$\frac{K}{s + K}$</td>
<td>$\frac{K(\tau_2 s + 1)}{\tau_1 s^2 + K(\tau_2 s + 1)}$</td>
<td>$\frac{K(\tau_2^2 + 2\tau_2 s + 1)}{\tau_1^2 s^3 + K(\tau_2^2 s^2 + 2\tau_2 s + 1)}$</td>
</tr>
<tr>
<td>$B_L$</td>
<td>$\frac{K^2}{4}$</td>
<td>$\frac{\tau_2^2 K + \tau_1}{4\tau_1 \tau_2}$</td>
<td>$\frac{\tau_2^2 K(2\tau_2 K + 3\tau_1^2)}{4\tau_1^2(2\tau_2 K - \tau_1^3)}$</td>
</tr>
<tr>
<td>Formulation II: tracking loop description in terms of natural frequency</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F(s)$</td>
<td>$\frac{\omega_n}{K}$</td>
<td>$\frac{a\omega_n s + \omega_n^2}{K s}$</td>
<td>$\frac{b\omega_n s^2 + a\omega_n^3 s + \omega_n^3}{K s^3}$</td>
</tr>
<tr>
<td>$H(s)$</td>
<td>$\frac{\omega_n}{s + \omega_n}$</td>
<td>$\frac{a\omega_n s + \omega_n^2}{s^2 + a\omega_n s + \omega_n^2}$</td>
<td>$\frac{b\omega_n s^2 + a\omega_n^3 s + \omega_n^3}{s^3 + b\omega_n s^2 + a\omega_n^3 s + \omega_n^3}$</td>
</tr>
<tr>
<td>$B_L$</td>
<td>$\frac{\omega_n}{4}$</td>
<td>$\frac{a^2 s^2 + 1}{4a - \omega_n}$</td>
<td>$\frac{a b^2 + a^2 - b}{4(ab - 1)} \omega_n$</td>
</tr>
<tr>
<td>Relationship between above two formulations</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega_n = K = K_1(K_2)$</td>
<td>$a\omega_n = \frac{\tau_2}{\tau_1} K$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b\omega_n = \left(\frac{\tau_2}{\tau_1}\right)^2 K$</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$\omega_n^3 = \frac{1}{\tau_1} K$</td>
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<tr>
<td>$\omega_n^3 = \frac{1}{\tau_1} K$</td>
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</tr>
</tbody>
</table>

Fig. 4 Loop filters, closed-loop transfer functions, and equivalent noise bandwidths $B_L$ (Hz) for different orders of receiver tracking loops

error for a second-order loop with constant velocity input and is optimal for constant acceleration input. The limitation exists during the transient period when going from no acceleration to constant acceleration.

From eqn. 9 the error transfer functions due to user dynamics are

\[
\frac{\theta_e(s)}{\theta_i(s)} = \frac{s}{s + \omega_n} \quad \text{(first order)} \quad \text{(12a)}
\]

\[
= \frac{s^2}{s^2 + a\omega_n s + \omega_n^2} \quad \text{(second order)} \quad \text{(12b)}
\]

\[
= \frac{s^3}{s^3 + b\omega_n s^2 + a\omega_n^2 s + \omega_n^3} \quad \text{(third order)} \quad \text{(12c)}
\]

and the dynamics stress errors are determined as follows.

3.2.1 Code tracking: Peak errors caused by a range rate, range acceleration, and jerk step in the third-order loop can be approximated by the steady-state response of a first-, second-, and third-order loop caused by a range rate, range acceleration, and jerk step function, respectively. The three-sigma steady-state errors can be written as (in units of chips)

\[
R_e = \frac{\tau_1}{\omega_n} \quad \text{(first order)} \quad \text{(13a)}
\]

\[
= \frac{\tau_1}{\omega_n^2} \quad \text{(second order)} \quad \text{(13b)}
\]

\[
= \frac{\tau_1}{\omega_n^3} \quad \text{(third order)} \quad \text{(13c)}
\]

where $d^n\tau/dt^n$, $n = 1 \ldots 3$, is the maximum line-of-sight jerk dynamics. For example, $d^3\tau/dt^3$ represents the maximum line-of-sight jerk dynamics. For a dynamic user platform, the closed-loop bandwidth must be sufficiently wide so as to track the transient variation of delay against time that is
have lost lock. The peak errors caused by a phase rate (frequency), phase acceleration (frequency rate), and phase jerk (frequency acceleration) step in the loop can be approximated by the steady-state response of a first-, second-, or third-order loop, respectively, and the steady-state errors are

\[
\phi_e = \frac{\Delta f}{\omega_p} \quad \text{(first order)} \tag{14a}
\]

\[
= \frac{\Delta f}{\omega_a} \quad \text{(second order)} \tag{14b}
\]

\[
= \frac{\Delta f}{\omega_j} \quad \text{(third order)} \tag{14c}
\]

where \(\Delta f\) and \(\phi_e\) are in units of Hz and cycles. From the rule-of-thumb \([4, 5]\), the three-sigma code noise should not exceed 1/2 chip of the chip length and the carrier noise should not exceed 45° of the wavelength for maintaining signal lock. Therefore the one-sigma code noise should not exceed 1/6 chip of the chip length and the carrier noise should not exceed 15° of the wavelength, i.e.

\[
\sigma_{\text{DLL}} = \sigma_{\text{PLL}} + \frac{\phi_e}{3} \leq \frac{d}{6} \quad \text{(chips)} \tag{15}
\]

and

\[
\sigma_{\text{PLL}} = \sqrt{\sigma_{\text{DLL}}^2 + \sigma_i^2 + \sigma_a^2} + \frac{\phi_e}{3} \leq 15 \quad \text{(degrees)} \tag{16}
\]

for DLL and PLL, respectively. The parameters \(\sigma_i\) and \(\sigma_a\) are the one-sigma vibration induced oscillator jitter and Allan variance-induced oscillator jitter, respectively, which are small when compared with the other two and are negligible without loss of generality.

### 4 Optimal and suboptimal linear solutions

When designing a system, there may be many possible results depending on the criteria of performance, the nature of the input signal, and restrictions placed on loop configuration \([7]\). There is usually no unique optimal result that applies under all conditions. Improvement in one area of performance is usually at the expense of degrading the other and therefore some compromise between the two is always necessary.

The term ‘optimal’ here is meant the one with minimum error variance. Linear approximation is made when implementing the analysis to the tracking loop. In addition, the noise is assumed white. The propagation of the error for a Kalman filter can be described by the Riccati equation

\[
P = FP + PF^T + GQG^T - PHR^{-1}HP \tag{17}
\]

When the system reaches steady state, \(P = 0\), the equation becomes an algebraic Riccati equation (ARE), which can be solved for the steady-state minimum covariance matrix and then the optimal Kalman gain matrix

\[
K_{\infty} = P_{\infty}H^TR^{-1} \tag{18}
\]

The steady-state Kalman filter uses gains derived from the steady-state covariance and provides suboptimal solutions.

The example used for illustration is for an airplane that rolls into a turn or a step input is applied to the elevator controls, adopted from \([6]\). Such a manoeuvre may be modelled as a ramp in acceleration, starting at zero acceleration at time \(t_0\) and levelling off at constant value acceleration \(A\) at time \(t_f\). This profile is described in terms of the acceleration derivative as a pulse in jerk,
\[ t_1 - t_0 \text{ wide and with amplitude } A/(t_1 - t_0). \] The acceleration profile for a high performance aircraft subjected to a step stick elevator input at Mach 0.8 is very close to a ramp levelling off after 0.6 s at 6 G acceleration. This manoeuvre is typical for a high performance jet aircraft operating in a terrain-following mode. For such a case, the parameter jerk \( J = 10 \text{ G/s} = 98.1 \text{ m/s}^3 \) is used as the dynamic input to the tracking loop.

### 4.1 Typical optimal gain and variance profiles

The time-varying error covariance and Kalman gain matrix can be obtained using the fourth-order Runge-Kutta integrator. To observe the typical gain and variance profiles, assume that the signal power condition is at \( C/N_0 = 30 \text{ dB-Hz} \), the dynamic equation is modelled as a 10 G/s jerk motion corrupted by white noise i.e. \( x = 98.1 + w \) where \( w \sim N(0, Q) \) and \( Q = 10 \), and the initial covariance is

\[
P_0 = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 0 & 1000 \end{bmatrix}
\]

Based on these assumptions the optimal bandwidth \( B_L = 12.2528 \text{ Hz} \), and consequently, the minimum measurement noise \( R = 105.2581 \text{ m}^2 \) are obtained (explained in Section 5). Fig. 7 shows the typical profiles of Kalman gains and error variances for a third-order GPS receiver tracking loops.

**Fig. 7** Typical profiles for third-order tracking loops  
*a* Kalman gains  
*b* Error variance

### 4.2 Suboptimal solutions

Referring to Fig. 3, the transfer functions in terms of original circuit parameters can be represented in terms of natural frequency, which provides better insight on the relationship between \( B_L \) and \( \omega_n \). The suboptimal solutions and related properties are described as follows.

#### 4.2.1 First order:

Both the general and optimal forms of the transfer function and bandwidth have the same result

\[
H(s) = \frac{K_1}{s + K_1} = \frac{\omega_n}{s + \omega_n}; \quad B_L = \frac{\omega_n}{4}
\]

The minimum variance and Kalman gain, respectively, are

\[
P_\infty = (QR)^{1/2}; \quad K_\infty = \frac{K_1}{(QR)^{1/2}} = \omega_n
\]

\[
\omega_n = (QR)^{1/2}
\]

#### 4.2.2 Second order:

The general form of transfer function and bandwidth for the second-order loop are

\[
H(s) = \frac{K_1 s + K_2}{s^2 + K_1 s + K_2} = \frac{\omega_n s + \omega_n^2}{s^2 + \omega_n s + \omega_n^2}; \quad B_L = \frac{a^2 + 1}{4a} \omega_n
\]

respectively. The minimum covariance and Kalman gain matrices, respectively, are

\[
P_\infty = \begin{bmatrix} \sqrt{2Q^{1/4}R^{1/4}} & Q^{1/2}R^{1/2} \\ Q^{1/2}R^{1/2} & \sqrt{2Q^{1/4}R^{1/4}} \end{bmatrix};
\]

\[
K_\infty = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \frac{\sqrt{2}(Q/R)^{1/4}}{\omega_n}; \quad \omega_n = (Q/R)^{1/4}
\]

The performance is optimal when

\[
H(s) = \frac{\sqrt{2\omega_n s + \omega_n^2}}{s^2 + \sqrt{2\omega_n s + \omega_n^2}}
\]

resulting in the bandwidth \( B_L = 3\omega_n/4\sqrt{2} \).

#### 4.2.3 Third order:

The general forms of the transfer function and bandwidth for the third-order loop are

\[
H(s) = \frac{K_1 s^2 + K_3 s + K_5}{s^3 + K_1 s^2 + K_2 s + K_3} = \frac{\omega_3 s^2 + \omega_3 s + \omega_3}{s^3 + \omega_3 s^2 + \omega_3 s + \omega_n}; \quad B_L = \frac{ab^2 + a^2 - b}{4(ab - 1)} \omega_n
\]

respectively. The minimum covariance and Kalman gain matrices, respectively, are

\[
P_\infty = \begin{bmatrix} 2(Q/R)^{3/6} & 2(Q/R)^{1/3} & Q^{1/2}R^{1/2} \\ 2(Q/R)^{1/3} & 2(Q/R)^{1/2} & 2Q^{1/2}R^{1/2} \\ 2Q^{1/2}R^{1/2} & 2(Q/R)^{1/2} & 2Q^{1/2}R^{1/2} \end{bmatrix};
\]

\[
K_\infty = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 2(Q/R)^{1/6} \\ 2(Q/R)^{1/3} \\ (Q/R)^{1/2} \end{bmatrix} = \begin{bmatrix} 2\omega_n \\ 2\omega_n^2 \\ \omega_n \end{bmatrix}; \quad \omega_n = (Q/R)^{1/6}
\]

The performance is optimal when

\[
H(s) = \frac{2\omega_n s^2 + 2\omega_n s + \omega_n^3}{s^3 + 2\omega_n s^2 + 2\omega_n s + \omega_n^3}
\]

resulting in the bandwidth \( B_L = 5\omega_n/6 \).

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Based on the preceding discussion, the values of $B_L/\omega_n$ are

$$\frac{B_L}{\omega_n} = \frac{1}{4} \quad \text{(first order)}$$

$$= \frac{a^2 + 1}{4a} \quad \text{(second order)}$$

$$= \frac{ab^2 + a^2 - b}{4(ab - 1)} \quad \text{(third order)}$$

If defining the parameter $\alpha = B_L/\omega_n$, the dynamic stress errors with minimum mean square error can then be obtained utilising $\alpha$ and eqn. 14

$$\phi = \frac{\Delta f}{\omega_n} = 0.25 \frac{\Delta f}{B_L} \quad \text{(first order)} \quad (19a)$$

$$= \frac{\Delta f}{\omega_n} = \left(\frac{3}{4}\right)^2 \frac{\Delta f}{B_L} \approx 0.2812 \frac{\Delta f}{B_L} \quad \text{(second order)} \quad (19b)$$

$$= \frac{\Delta f}{\omega_n} = \left(\frac{5}{6}\right)^3 \frac{\Delta f}{B_L} \approx 0.5787 \frac{\Delta f}{B_L} \quad \text{(third order)} \quad (19c)$$

5 Optimisation of loop bandwidth

Optimal bandwidth is a function of the input carrier-to-noise ratio. The variable-gain loop is capable of performing an explicit estimate of $C/\text{No}$ as a criterion for adjusting the loop bandwidths to determine an optimal bandwidth that minimises the total loop tracking error

$$\sigma\text{DLL} = \frac{B_L}{2c/\text{No}} \left( 1 + \frac{2}{Tc/\text{No}} \right) \lambda_c + \frac{\alpha^2}{3} \frac{\tau(\alpha)}{B_L^2} \quad (20)$$

where $\tau(\alpha) = \alpha^2 \tau/d\alpha$. To minimise the total error the loop is designed to be capable of measuring $C/\text{No}$ and adjusting its bandwidth for optimal performance. It is seen that total DLL tracking error is a function of several parameters: $\sigma_{\text{DLL}} = f(B_L, C/\text{No}, T, \tau(\alpha))$, and $B_L$ value has opposite influence on $\sigma_{\text{DLL}}$. The optimal bandwidth is determined by differentiating $\sigma_{\text{DLL}}$ with respect to $B_L$ and setting the result equal to zero

$$\frac{\partial \sigma_{\text{DLL}}}{\partial B_L} = 0 \quad (21)$$

which yields

$$-\frac{1}{2c/\text{No}} \left( 1 + \frac{2}{Tc/\text{No}} \right) \lambda_c^2 \left( \frac{2\alpha^2}{3} \frac{\tau(\alpha)}{B_L^2} \right) B_L^{-(2n+1)} = 0 \quad (22)$$

The optimal loop bandwidth is found to be

$$(B_L)_{\text{DLL, optimum}} = \frac{\left( \frac{2\alpha^2}{3} \frac{\tau(\alpha)}{B_L^2} \right)^{2n+1}}{2c/\text{No}} \left( 1 + \frac{2}{Tc/\text{No}} \right) \lambda_c^2 \quad (23)$$
6 Sensitivity analysis

Choice of optimal bandwidth strongly depends on modulation parameters; it is \textit{a priori} knowledge of message statistics that permits threshold reduction. The adaptive loop performs selection of the desired bandwidth from a table of precomputed values based on the noise level $C/N_0$ estimate, and control logic to modify the bandwidths and prevent transients. As shown before, received $C/N_0$ is the key parameter in system performance analysis. In some situation, there is an incorrect estimate of noise level; in some applications, there is no need to adjust the loop to attain exactly the best performance. The sensitivity analysis usually involves in how the incorrect parameters (departure from the design point) selected influence on the error growth.

The filter gain in the error covariance eqn. 17 does not appear explicitly. Another form of error covariance propagation is described now. The error covariance relationships for a discrete filter with the same structure as the Kalman filter, but with an arbitrary gain matrix are written as

$$
P_k = (I - K_k H_k)P_k^{\text{old}}(I - K_k H_k)^T + KRK^T$$

(27)

$$
P_{k+1} = \Phi_k P_k \Phi_k^T + Q_k$$

(28)

which can be described in a single differential equation for the continuous filter

$$
\dot{P} = (F - KH)P + P(F - KH)^T + GQG^T + KRK^T$$

(29)

This equation defines the error covariance for the filter with a general filter gain matrix $K$, which can be solved for the covariance of an arbitrary gain model. The sensitivity analysis can be conveniently implemented by using this representation. Taking the partial derivative of $P_\infty$ with respect to $K$, using $\partial P_\infty / \partial K = 0$ for a minimum gives the same result as eqn. 17.

If the fixed-gain matrix $K$ of a filter has been designed for particular values of $Q$ and $R$, the steady-state error covariance will vary linearly with the actual process noise spectral density or measurement error spectral density. If the actual noise variances are assumed fixed and the design values of $Q$ and $R$ are varied, quite different curves result. Any deviation of the design variances, and consequently $K$, from the correct values will cause an increase in the filter error variance (a consequence of the optimality of the filter). More information on sensitivity analysis is contained in [11].

The error covariance for the filter with arbitrary gain model, can be obtained by using eqn. 29.

First order

$$
\dot{P} = -2KP + K^2R + Q
$$

(30)

Second order

$$
\begin{align*}
\dot{P}_{11} &= -2K_{11}P_{11} + 2P_{12} + K_{12}R \\
\dot{P}_{21} &= -K_{21}P_{11} - K_{12}P_{12} + P_{22} + K_{21}K_{12}R \\
\dot{P}_{22} &= -2K_{22}P_{12} + K_{22}^2R + Q
\end{align*}
$$

(31)
By setting $P = 0$, the solutions are found to be

$$P_{11} = -2K_1P_{11} + 2P_{12} + K_1^2R;$$

$$P_{12} = P_{21} = -K_1P_{11} - K_1P_{12} + P_{22} + K_1K_2R$$

$$P_{13} = P_{31} = -K_1P_{11} - K_1P_{13} + P_{32} + K_1K_3R;$$

$$P_{22} = -2K_2P_{12} + 2P_{23} + K_2^2R$$

$$P_{23} = P_{32} = -K_2P_{12} - K_2P_{13} + P_{33} + K_2K_3R;$$

$$P_{33} = -2K_3P_{13} + K_3^2R + Q$$

(32)

By setting $P = 0$, the solutions are found to be

$$P_{\infty} = \frac{Q + K_2^2R}{2K} \quad \text{(first order)} \quad \text{(33)}$$

$$= \frac{1}{2K, K_2} \begin{bmatrix}
Q + K_2^2K_2R + K_2^2R & (Q + K_2^2R)K_1 \\
(Q + K_2^2R)K_1 & K_2^2Q + K_2Q + K_2^2R
\end{bmatrix} \quad \text{(second order)} \quad \text{(34)}$$

$$= \frac{1}{\Delta} \begin{bmatrix}
P_{11} & P_{12} & P_{13} \\
P_{22} & P_{23} & P_{33}
\end{bmatrix}_{\infty} \quad \text{(third order)} \quad \text{(35)}$$

where

$$P_{11\infty} = K_1^2K_2K_3R - K_1K_2^2R + K_2^2K_3Q + K_1Q$$

$$P_{12\infty} = K_1(K_2^2K_3R + K_2Q)$$

$$P_{13\infty} = (K_2^2R + Q)(K_2K_3 - K_3)$$

$$P_{22\infty} = K_2^3K_2R + K_2^2Q + K_2^2R + K_2Q$$

$$P_{23\infty} = K_3(K_2K_3^2R + K_3^2Q)$$

$$P_{33\infty} = K_2K_3^2R + K_2^2K_3Q + K_1K_2^2Q - K_2K_3Q$$

$$\Delta = 2K_3(K_2K_3 - K_3)$$

The sensitivity curve of the error growth is predicted using eqns. 33–35 for first- to third-order loops, respectively. The terms ‘sensitivity’ here involves the error growth due to incorrect estimate of the $C/N_0$ (i.e. design $C/N_0$ deviating from the actual ratio). The optimal bandwidth $(B)_{\text{DLL,optimum}}$ is first determined by a given estimate of $C/N_0$. With knowledge of $C/N_0$ and the related optimal bandwidth, the measurement noise variance $R$, and consequently the Kalman gain matrix $K$, are determined. An incorrect estimate of $C/N_0$ gives an incorrect gain matrix and results in error growth.

The information flow chart for the sensitivity analysis is shown in Fig. 10. Procedures for theoretical approach are as follows. After input of design and actual $C/N_0$ and dynamics information, the optimal bandwidths for design and actual environments, i.e. $(B)_{\text{design}}$, and $(B)_{\text{true}}$, are subsequently determined by eqns. 24 (for DLL) and 26 (for PLL). Once the optimal bandwidths are determined, noise information, i.e. $R_{\text{design}}$ and $R_{\text{true}}$, can be estimated from eqns. 7 (for DLL) and 8 (for PLL). Consequently the optimal gain matrices, i.e. $K_{\text{design}}$ and $K_{\text{true}}$, are determined by eqns. 17 and 18. Finally, the error covariance is determined by eqns. 33 and 35. Error variance obtained by actual measurement noise $R_{\text{true}}$ with $K_{\text{true}}$ (based on correct estimate on $C/N_0$) has a minimum value while error variance obtained by actual measurement noise $R_{\text{true}}$ with $K_{\text{true}}$ (based on incorrect estimate on $C/N_0$) has a minimum value.
\( K_{\text{design}} \) (due to originally incorrect estimate on \( C/N_0 \)) results in error growth. The numerical simulation essentially follows the same procedures as theoretical approach except that random numbers generated by Matlab are employed. The error variances are determined by analysing the output data from the filter. Figs. 11 and 12 give example on the sensitivity for the third-order DLL (C/A code) and PLL (\( f_c \) carrier), respectively. The theoretical results, represented by solid lines, and the simulation results, represented by symbols, are in very good agreement.

7 Conclusions

Analysis of optimisation and sensitivity of GPS receiver tracking loops has been presented. The maximum bandwidth threshold is essentially governed by the thermal noise jitter; however, the minimum bandwidth threshold is mainly governed by the dynamic stress error. Tracking loops are considered without internal or external aiding such that the influence of the dynamics stress error component on the total tracking error is significant. A theoretical approach for predicting optimal loop bandwidth has been derived. Sensitivity analysis of the error growth due to an incorrect estimate of the carrier-to-noise ratio was conducted. Mathematical derivation of theoretical results of the tracking loop performance, including optimisation and sensitivity, has been performed from first to third orders. Numerical examples on third-order loops for high dynamics applications have been provided. Both theoretical and simulation results are in very good agreement. Implementation on the other orders of loops can be done without difficulty based on the same procedure.

8 References